Lattice Radial Quantization.

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Abstract

We regularize conformal field theories in radial quantization using lattice techniques. Only angular separations matter. Scale is logarithmically discretized in equal intervals. As an application, we set out to compute the critical exponent η for the 3D Ising model and present some preliminary results.

Outline

Beautiful theories

Conformal theories

Foliations

The icosahedral Transfer matrix

Spectrum

Regularization

Velocity of light renormalization

Preliminary numerical results

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Being a snob

"Beauty" $= \frac{\text{nr. of nontrivial results}}{\text{nr. of free continuos parameters}}$. To be worthwhile of our time, a beautiful theory ought to be relevant to Nature. Such beautiful theories never are exactly soluble, nor are they completely perturbative. The top examples for particle physicists are

- The most relevant beautiful particle theory is QCD with a moderate number of massless quarks.
- The most beautiful unparticle theory is $SU(N)$ gauge theory with massless fermionic matter in the conformal window. Maybe it is relevant to Nature.

QCD is essentially a single scale theory, and lattice techniques work and are in principle exact. Walking technicolor type of theories are harder because they have too wide a range of scales for an ordinary lattice approach. One can make progress however by considering their conformal cousins. Of primary importance are the anomalous dimensions of some composite scaling fields in the IR.

Outline

Main point and application

In a conformal theory there is no scale and only angles matter. Rather than uniformly discretizing scales one should uniformly discretize their logarithm as per AMR logic. Simulating a conformal theory by uniformly discretizing scales might be impractical. What to do in the “almost conformal” case is another topic.

There is no point in further generalities. I’ll focus on the critical 3D Ising model because one can simulate it very efficiently using cluster algorithms. The objective would be to exploit it being a CFT in order to extract the anomalous dimension of the magnetization operator. Our numerical results are preliminary.

Outline

Radial quantization and its cousins

In radial quantization one views R^3 as a sequence of 2D concentric spherical shells whose radial density is uniform in the logarithm of the radius measured from the common center. The coordinates on the shells are angles. By a conformal transformation flat R^3 gets mapped into an infinite cylinder raised on top of a 2D sphere. One quantizes by defining a transfer matrix along the cylinder. The logarithm of the eigenvalues of this transfer matrix provide the spectrum of dimensions of all operators.
We could foliate R^3 in other ways. We choose to foliate it into concentric 2D icosahedral shells, again at uniform density in the logarithm of the distance from the common center. The shells have 20 flat equilateral triangular faces and 12 corners. One can define now an icosahedral transfer matrix.

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Spectrum and states

The icosahedral transfer matrix T has the same spectrum as the radial one: it consists of the exponent of all the eigenvalues of the dilatation operator in the Wilson-Fisher CFT.
The eigenstates of T are different from the radial case. Most spectral regularities would seem accidental because the $SO(3)$ invariance is hidden. Regularities reflecting the discrete symmetry group $\mathbf{I}$ of the icosahedron remain evident. A multiplet of angular momentum l under $SO(3)$ would transform irreducibly under $\mathbf{I}$ if $l = 0, 1, 2$ because the icosahedron has 3-fold and 5-fold symmetry axes. The remaining 2 irreps of $\mathbf{I}$ appear in the decomposition of $l = 3$; their degeneracy would seem accidental if we did not know about the hidden $SO(3)$.
Let $M(x)$ be the exact scaling field corresponding to the highest eigenvalue of T in the Z_2-odd sector. One can construct out of it 5 orthogonal multiplets transforming irreducibly under $\mathbf{I}$ in representations subduced from the $l = 1, 2, 3$ irreps of $SO(3)$.

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Descendants

Let m_0 be the dimension of $M(x)$ and m_l the dimensions of the descendants constructed from $M(x)$ for $l = 0, 1, 2$. One has

$$m_l = m_0 + l$$

At this point we went beyond just using scaling, which reflects only dilatation invariance. The integer spacing is a consequences of full conformal invariance. Our regularization preserves (if present in the continuum) the crucial discrete inversion transformation which extends the symmetry in the flat case to the full conformal group.

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The lattice

Each equilateral triangle is replaced by a fine triangular mesh defined by adding $s-1$ extra points at equal distance on each icosahedral edge. The total number of sites per shell is $10s^2+2$. Our UV cutoff Λ will be $\propto s$ for large s because the size of a shell is some fixed number in terms of angular extent. The UV cutoff is dimensionless in radial quantization.
On each site we place an Ising spin. To all intra- and inter- links connecting sites i and j we attach the standard weight

$$e^{b\sigma_i\sigma_j}$$

$b>0$ is the Ising coupling and needs to be tuned to b_c to get into the domain of attraction of the Wilson Fisher fixed point. The SW cluster update algorithm only needs the abstract graph of the structure in order to proceed.

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Fixing homogeneity

Except at the corners, the shells are locally a flat triangular lattice. Neighboring shells are connected chain like. There is local regularity, but the approximate local discrete rotational invariance around a vertex does no mix in-shell with intra-shell directions.

For a given s, the eigenvalues corresponding to m_l are μ_l and the near shell separation is $\Delta\tau = 1$. The continuum shell separation is Δt. We define κ so that $m_l = \kappa\mu_l\Lambda$. Then, $t = \frac{\tau}{\Lambda}$ to ensure $m_l t = \kappa\mu_l\tau$.

We can extract $\kappa\Lambda$ from

$$\mu_l' - \mu_l = \frac{l' - l}{\kappa\Lambda}$$

Then we can extract m_0 from $m_0 = \kappa\Lambda\mu_0$. The anomalous dimension of $M(x)$, η is defined by

$$m_0 = \frac{1 + \eta}{2}$$

Outline

The refinement level is s and the number of shells is $16s$; at $b = b_c$, $\kappa\Lambda \propto s$ must hold to leading order in s. $b = 0.16107$.

b	s	$\frac{\mu_2-\mu_1}{\mu_1-\mu_0}$	$\frac{s}{\kappa\Lambda}$	η
0.16107	6	0.966(5)	0.724(2)	0.084(2)
0.16107	7	0.966(5)	0.728(1)	0.066(2)
0.16107	8	0.958(2)	0.7288(2)	0.052(1)
0.16107	9	0.954(2)	0.7311(2)	0.036(1)
0.16107	10	0.957(2)	0.7298(2)	0.044(1)
0.16107	11	0.961(2)	0.7267(2)	0.064(1)
0.16107	12	0.973(2)	0.7181(2)	0.123(1)

Previous simulations, using cubic shells and a different approach, gave, after extrapolation to infinite cutoff, $\eta = 0.002 \pm 0.010$; the expected value is $\eta \approx 0.036$.
Maybe we found the correct value of η at $s = 9$ not by accident.

"Masses" or more correctly dimensions

We probably are not at $b = b_c(\infty)$. Something is happening at $s = 9$. Maybe we are in the would-be Z_2 broken side of $b_c(\infty)$. Maybe periodic boundary conditions are better. So far, our analysis has been rough.